Taylor-Galerkin Residual Distribution Schemes
with Application to Astrophysical Flows

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REU Student: Matt Elsey, U. Michigan – developed grid generator in C++
Astrophysical fluids: stellar collapse, interaction of BHs, BH accretion, 

Numerical challenges: shocks, geometric singularities, constraints, 

Numerical methods: FD, spectral, SPH, HRSC (Godunov, ENO, & central) 

Goal of this research: develop alternative based on residual distribution 

Properties: unstructured & truly multi-D $\rightarrow$ high-order & unsteady under research 

This talk: basic idea & preliminary results on simple equations
Residual distribution schemes

Basic idea: [Roe, 1987], [Deconinck et al., 1993]
- Hyperbolic balance law: \( \partial_t q + \nabla \cdot \vec{F} = \psi \)
- Solution is stored on nodes of a triangular (tetrahedral) mesh
- Create residual in each element & distribute to nodes:
  \[
  \phi^T \approx \iint_T \left[ \nabla \cdot \vec{F} - \psi \right] dA \quad \implies \quad \phi_1^T, \phi_2^T, \phi_3^T
  \]
- Update solution by collecting all residuals that influence node \( i \):
  \[
  Q_i^{n+1} = Q_i^n - \Delta t \frac{\sum_{T:i \in T} \phi_i^T}{|C_i|}
  \]

Why RD schemes for astrophysics?
- Shock-capturing, high-order on smooth flows
- Can handle complicated, even time-dependent geometry
- RD schemes are well-balanced \( \implies \) steady-states are accurately preserved
1D Residual distribution schemes

Note: For $q_t + f(q)_x = 0 \implies$ RD schemes $\equiv$ HRSC (Roe’s method)

Distribution point interpretation of limiters: $(q_t + uq_x = 0)$

$$\phi_1 = \left( \frac{x_i - p}{x_i - x_{i-1}} \right) \phi^T \quad \text{and} \quad \phi_2 = \left( \frac{p - x_{i-1}}{x_i - x_{i-1}} \right) \phi^T,$$

where $p \in [x_{i-1}, x_i]$

- **N-Scheme:** upwind method is “narrow” $\implies$ either $p = x_i$ or $p = x_{i-1}$
- **Lax-Wendroff:** $p = x_{i-1/2} + \frac{1}{2} \Delta x \nu$, where $-1 \leq \nu = \frac{u\Delta t}{\Delta x} \leq 1$
- **Limiters:** Find $p$ s.t. solution in non-oscillatory $\implies$ if $u > 0$: $p_{LxW} \leq p \leq p_N$

Higher-order for 1D steady-states:

- Create sub-elements in each element, solution stored at each node

\[ \phi^T = f_i - f_{i-1} - \int_{x_{i-1}}^{x_i} \psi \, dx \]

- In each element create interpolant of $\psi$, integrate exactly in each sub-element
Example: 1D periodic advection

$q(x,t) \text{ at } t = 10 - \text{RedPack} - [N\text{-Scheme}]$

$q(x,t) \text{ at } t = 10 - \text{RedPack} - [LxW]$
Application: relativistic gas dynamics

\[
\frac{\partial}{\partial x^\sigma} \left( \sqrt{-g} \begin{bmatrix} D \\ S^j \\ E \end{bmatrix} \right) + \frac{\partial}{\partial x^i} \left( \sqrt{-g} \begin{bmatrix} \rho v^i W \\ \rho h v^i W^2 + p g^i_{i0} \end{bmatrix} \right) = - \begin{bmatrix} 0 \\ \sqrt{-g} \Gamma_{i\mu\lambda}^i T^{\mu\lambda} \\ \sqrt{-g} \Gamma^0_{\mu\lambda} T^{\mu\lambda} \end{bmatrix}
\]

Covariant formulation [Papadopolous & Font, 2000]

- \( q = (D, S^j, E) \equiv (\text{rest-mass, momentum, energy}) \)
- \( u = (\rho, v^j, p) \equiv (\text{density, fluid 3-velocity, fluid pressure}) \)
- Specific relativistic enthalpy: \( h = 1 + \frac{p \Gamma}{\rho (\Gamma - 1)}, \quad \Gamma \equiv \text{gas constant} \)
- Relationship between conserved and primitive variables

\[
\begin{bmatrix} D \\ S^j \\ E \end{bmatrix} = \begin{bmatrix} \rho W \\ \rho h v^j W^2 + p g^0_{i0} \\ \rho h W^2 + p g^{00} \end{bmatrix}, \quad W = \frac{1}{\sqrt{-g_{00} - 2g_{0i} v^i - g_{ij} v^i v^j}}
\]

Choice of spacetime foliation

- Spacelike foliations of spacetime \((g^{00} \neq 0) \Rightarrow \text{Newton iteration for } W\)
- Null foliations of spacetime \((g^{00} = 0) \Rightarrow \text{no Newton iteration required}\)
**Example 1:** Minkowski coordinates \((t, x) = (x^0, x^1)\),
\[
ds^2 = -dx^0 dx^0 + dx^1 dx^1
\]

**Example 2:** Null coordinates \((t, x) = (x^0 - x^1, x^1)\),
\[
ds^2 = -dx^0 dx^0 + 2dx^0 dx^1
\]
Example: radial dust accretion

Eddington-Finkelstein coordinates:

\[ ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \frac{4M}{r} dt\, dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2 \]

- Start from constant density state with \( p \approx 0 \), run to steady-state
- Exact solution known, experimental convergence rate: 2, 4, 4, and 6
Example: radial dust accretion

Null Eddington-Finkelstein coordinates:

$$ds^2 = - \left(1 - \frac{2M}{r}\right) d\hat{t}^2 + 2 \hat{t} dr + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

- Start from constant density state with $p \approx 0$, run to steady-state
- Exact solution known
2D N-scheme

- Narrow scheme: minimal stencil for 1st order, 2D upwind method
- Residual: \( \phi_i = \frac{1}{2} [\bar{u} \cdot \bar{n}_i]^+ (Q_i - \bar{Q}) \equiv \beta_i \phi^T \)
- Conservation: \( \bar{Q} \) chosen so that \( \sum_i \phi_i = \phi^T \Rightarrow \sum_i \beta_i = 1 \)
- Monotonicity: N-scheme can be written as \( \phi_i = \sum_i \sum_j c_{ij} (Q_i - Q_j), \ c_{ij} \geq 0 \)
- Linear Preserving: improved accuracy in steady-state with \( \beta_j \rightarrow \frac{\beta_j^+}{\sum_j \beta_j^+} \)
2D High-order N-scheme

- [Abgrall & Roe, 2003] – high-order in steady-state using above elements
- \( P_k(x, y) \equiv \) interpolant over full element
- Construct a high-order residual by integrating \( P_k(x, y) \) in each sub-element \( T \):
  \[
  \phi^T = \iint_T P_k(x, y) \, dA \approx \iint_T \left[ \vec{\nabla} \cdot \vec{F} - \psi \right] \, dA
  \]
- In practice this is carried out using 2D Gaussian quadrature
- Distribute via the N-scheme in each sub-element
- To achieve high-order need to again apply: \( \beta_j \rightarrow \frac{\beta_j^+}{\sum_j \beta_j^+} \)
2D Lax-Wendroff scheme and limiting

- [Hubbard & Roe, 2000] – Limiters on N-scheme + Lax-Wendroff

- Lax-Wendroff method:
  \[
  \phi_i = \left( \frac{1}{3} + \frac{\Delta t}{4S^T} \left[ \bar{u} \cdot \bar{n}_i \right] \right) \phi^T
  \]

- Limiting: N-scheme is monotone, so don’t allow values to exceed N-scheme values

- Limiting for systems: open problem

- Higher-order: open, work by [Abgrall et al., 2005] and [Hubbard & Laird, 2005]
Example: 2D steady advection (smooth)

Numerical Grid

Relative L2-norm error (Error - h^p)

Radial scatter plot at t = 2 - RedPack - h = 4.0e-02

Radial scatter plot at t = 4 - RedPack - h = 4.0e-02
Example: 2D steady advection (discontinuous)
Example: 2D unsteady advection

\[ q(x,y,t) \text{ at } t = 0 \quad - \quad \text{RedPack} \quad - \quad h = 2.0 \times 10^{-2} \]

\[ q(x,y,t) \text{ at } t = 1 \quad - \quad \text{RedPack} \quad - \quad h = 2.0 \times 10^{-2} \]

\[ q(x,y,t) \text{ at } t = 1 \quad - \quad \text{CLAWPACK} \quad - \quad h = 2.0 \times 10^{-2} \]
Future work

1. Systems of conservation laws
2. Improve time accuracy
3. Develop 3D code (3D mesh generation already operational)
4. Relativistic Euler & magnetohydrodynamics
5. Simulation of black hole accretion
6. Dynamically evolving spacetimes